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Ergodic Projection for Quantum Dynamical Semigroups

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This paper deals with problems concerning ergodic projection for semigroups of bounded linear mappings on a W^*-algebra. A criterion for the normality of this projection is given and the structure of the fixed-point space is analyzed.

INTRODUCTION

This paper is devoted to studying various properties of ergodic projection for a semigroup $(\alpha_g: g \in \mathbb{G})$ of bounded linear mappings on a W^*-algebra M. We investigate the structure of the fixed-point space, and give a criterion for an ergodic projection to be normal. It is worth noting that the approach we adopt allows us to disregard any continuity properties of the action of $\mathbb{G}$ on M.

1. PRELIMINARIES AND NOTATION

Throughout the paper, M will stand for a W^*-algebra with predual M_* and identity $\mathbf{1}$. Let $\mathbb{G}$ be a semigroup and let α be a representation of $\mathbb{G}$ into the set of linear bounded mappings on M. A bounded linear functional φ on M is said to be $\mathbb{G}$-invariant if $\varphi \circ \alpha_g = \varphi$ for all $g \in \mathbb{G}$. We write $M^{*\mathbb{G}}$ (resp. $M_*^{\mathbb{G}}$) for the set of all $\mathbb{G}$-invariant (resp. normal $\mathbb{G}$-invariant) bounded linear functionals on M, and $M^{\mathbb{G}}$ stands for the space of all $\mathbb{G}$-invariant elements of M, i.e., $M^{\mathbb{G}} = \{x \in M: \alpha_g(x) = x, g \in \mathbb{G}\}$.

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A bounded linear mapping ϵ on M is called an ergodic projection if

(i) $\epsilon(M) \subset M^{G}$, $\epsilon^2 = \epsilon$
(ii) $\epsilon \circ \alpha_g = \alpha_g \circ \epsilon$ for all $g \in \mathbb{G}$
(iii) $\epsilon \in \overline{\text{conv}\{\alpha_g: g \in \mathbb{G}\}}$

where the closure of the convex hull is taken in the point-σ-weak topology on the space $\mathbb{B}(M)$ of all bounded linear mappings on M, i.e., the topology given by the system of seminorms $\{\|\cdot\|_{\varphi,x}: \varphi \in M_*, x \in M\}$

$$\|\Phi\|_{\varphi,x} := |\varphi(\Phi x)|, \qquad \Phi \in \mathbb{B}(M)$$

It follows that $\epsilon(M) = M^{G}$.

Remark 1. If ϵ is an ergodic projection, then:

(a) For each $\varphi \in M_*^{G}$, $\varphi \circ \epsilon = \varphi$.
(b) For each $\varphi \in M^*$, $\varphi \circ \epsilon \in M^{*G}$.

Indeed, (a) is a consequence of (iii), and (b) follows immediately from (ii).

Remark 2. It can easily be shown that if an ergodic projection is normal, then it is unique.

The problem of the existence of an ergodic projection for semigroups of mappings on a W^*-algebra was considered on various levels of generality in (Frigerio, 1978; Frigerio and Verri, 1982; Kümmerer and Nagel, 1979; Thomsen, 1985; Watanabe, 1979). In this paper, taking existence for granted, we investigate its structure and properties.

2. ERGODIC PROJECTION AND FIXED POINTS

Let $\mathbb{G}$ be an arbitrary semigroup and let $(\alpha_g: g \in \mathbb{G})$ be a representation of $\mathbb{G}$ in $\mathbb{B}(M)$ such that all α_g's are positive. Following Evans and Høegh-Krohn (1978) and Frigerio and Verri (1982) (cf. also Groh, 1986), we define the recurrent projection p_r as

$$p_r = \sup\{s(\varphi): \varphi \in M_*^{G}, \varphi \geq 0\}$$

where $s(\varphi)$ stands for the support of a normal positive linear functional φ. An important property of p_r is given in the lemma below. For any $a \in M$, $\varphi \in M^*$, define $a\varphi \in M^*$ as $\varphi(\cdot a)$, $\varphi a \in M^*$ as $\varphi(a \cdot)$, and $a\varphi a \in M^*$ as $\varphi(a \cdot a)$.

Lemma 1. Let the α_g's be positive contractions. Then for each $\varphi \in M_*^{G}$, $\varphi = p_r\varphi = \varphi p_r$.

Proof. Clearly, by the definition of p_r, $\varphi = p_r\varphi = \varphi p_r$ for $\varphi \in M_*^G$, $\varphi \geq 0$. The result will follow if we show that M_*^G is linearly spanned by its positive elements. For any linear functional φ we put

$$\varphi^*(x) = \overline{\varphi(x^*)}, \qquad \mathrm{Re}\,\varphi = \frac{1}{2}(\varphi + \varphi^*), \qquad \mathrm{Im}\,\varphi = \frac{1}{2i}(\varphi - \varphi^*)$$

Then Re φ and Im φ are Hermitian and $\varphi = \mathrm{Re}\,\varphi + i\,\mathrm{Im}\,\varphi$. It is easily seen that if $\varphi \in M_*^G$, then Re φ and Im φ are in M_*^G, too. Assume now that φ is Hermitian, $\varphi \in M_*^G$, and let $\varphi = \varphi_+ - \varphi_-$ be its Jordan decomposition. We have

$$\varphi = \varphi_+ \circ \alpha_g - \varphi_- \circ \alpha_g$$

and

$$\|\varphi_+ \circ \alpha_g\| = \varphi_+(\alpha_g(\mathbf{1})) \leq \varphi_+(\mathbf{1}) = \|\varphi_+\|$$
$$\|\varphi_- \circ \alpha_g\| = \varphi_-(\alpha_g(\mathbf{1})) \leq \varphi_-(\mathbf{1}) = \|\varphi_-\|$$

Thus

$$\|\varphi\| = \|\varphi_+\| + \|\varphi_-\| \geq \|\varphi_+ \circ \alpha_g\| + \|\varphi_- \circ \alpha_g\|$$
$$\geq \|\varphi_+ \circ \alpha_g - \varphi_- \circ \alpha_g\| = \|\varphi\|$$

which shows that

$$\|\varphi_+ \circ \alpha_g\| = \|\varphi_+\|, \qquad \|\varphi_- \circ \alpha_g\| = \|\varphi_-\|$$

so

$$\|\varphi\| = \|\varphi_+ \circ \alpha_g\| + \|\varphi_- \circ \alpha_g\|$$

and the uniqueness of the Jordan decomposition gives

$$\varphi_+ \circ \alpha_g = \varphi_+, \qquad \varphi_- \circ \alpha_g = \varphi_-$$

Thus we have, for $\varphi \in M_*^G$,

$$\varphi = (\mathrm{Re}\,\varphi)_+ - (\mathrm{Re}\,\varphi)_- + i[(\mathrm{Im}\,\varphi)_+ - (\mathrm{Im}\,\varphi)_-]$$

and each of the four functionals in the above decomposition belongs to M_*^G, which proves the claim. ■

Let $\Phi \in \mathbb{B}(M)$ and let e be a nonzero projection in M. We say that e reduces Φ if $\Phi(eMe) \subset eMe$; analogously, e reduces Φ^* if $\Phi^*(eM_*e) \subset eM_*e$.

Lemma 2. Let $\Phi \in \mathbb{B}(M)$ be positive and let e be a nonzero projection in M. Put $e^\perp = \mathbf{1} - e$. The following conditions are equivalent:

(i) e reduces Φ^*.

(ii) $e\Phi(x)e = e\Phi(exe)e$, $x \in M$.
(iii) There is a $\gamma > 0$ such that $\Phi(e^\perp) \leq \gamma e^\perp$.
(iv) $e^\perp$ reduces Φ.

Proof. (i) $\Rightarrow$ (ii). For each $\varphi \in M^*$, $(e\varphi e) \in eM^*e$ and thus, by assumption, $\Phi^*(e\varphi e) \in eM^*e$, that is,

$$\Phi^*(e\varphi e) = e\Phi^*(e\varphi e)e$$

Consequently, for $x \in M$ we have

$$\begin{aligned}\varphi(e\Phi(x)e) &= (e\varphi e)(\Phi(x)) = \Phi^*(e\varphi e)(x) = (e\Phi^*(e\varphi e)e)(x)\\ &= \Phi^*(e\varphi e)(exe) = \varphi(e\Phi(exe)e)\end{aligned}$$

proving (ii).

(ii) $\Rightarrow$ (iii). Putting $x = \mathbf{1}$ in (ii), we get

$$e\Phi(\mathbf{1})e = e\Phi(e)e$$

Thus $e\Phi(e^\perp)e = 0$, which gives the equality

$$\Phi(e^\perp) = e^\perp\Phi(e^\perp)e^\perp$$

implying that $\Phi(e^\perp) \leq \gamma e^\perp$ for some $\gamma > 0$.

(iii) $\Rightarrow$ (i). Let $\varphi \in eM^*e$, $\varphi \geq 0$. We shall show that $\Phi^*(\varphi) = e\Phi^*(\varphi)e$, which means that $\Phi^*(\varphi) \in eM^*e$. Indeed, we have

$$\Phi^*(\varphi)(e^\perp) = \varphi(\Phi(e^\perp)) \leq \gamma\varphi(e^\perp) = 0$$

showing that the support of $\Phi^*(\varphi)$ considered as a normal functional in the algebra M^{**} is contained in e, and so $\Phi^*(\varphi) = e\Phi^*(\varphi)e$. Since the positive elements span eM^*e, the same is true for any $\varphi \in eM^*e$.

(iii) $\Leftrightarrow$ (iv). Obvious. ■

Let now the α_g's be positive contractions. By Thomsen (1985), Lemma 1, $\alpha_g(s(\varphi)) \geq s(\varphi)$ for each $\varphi \in M_*^{\mathbb{G}}$, $\varphi \geq 0$. Consequently,

$$\alpha_g(p_r) \geq p_r, \qquad g \in \mathbb{G} \tag{1}$$

and the contractivity yields

$$\alpha_g(p_r^\perp) \leq p_r^\perp, \qquad g \in \mathbb{G}$$

Since $\epsilon \in \overline{\text{conv}\{\alpha_g\colon g \in \mathbb{G}\}}$, we have also

$$\epsilon(p_r^\perp) \leq p_r^\perp$$

and, by virtue of Lemma 2, we get

$$p_r\alpha_g(x)p_r = p_r\alpha_g(p_rxp_r)p_r, \qquad g \in \mathbb{G}, \quad x \in M \tag{2}$$

$$p_r\epsilon(x)p_r = p_r\epsilon(p_rxp_r)p_r, \qquad x \in M \tag{3}$$

Following Frigerio and Verri (1982), we define mappings α_g^r from $p_r M p_r$ into itself by

$$\alpha_g^r(p_r x p_r) = p_r \alpha_g(x) p_r, \qquad x \in M$$

Taking into account (1) and (2), we infer that $(\alpha_g^r\colon g \in \mathbb{G})$ is a semigroup of linear positive identity-preserving mappings on $p_r M p_r$. From Lemma 1 and formula (2) it follows that the spaces $M_*^{\mathbb{G}}$ and $(p_r M p_r)_*^{\mathbb{G}}$—the space of normal (α_g^r)-invariant linear functionals on $p_r M p_r$—are isometrically isomorphic to each other with the isomorphism given by $M_*^{\mathbb{G}} \ni \varphi \to \varphi | p_r M p_r \in (p_r M p_r)_*^{\mathbb{G}}$. Moreover, $\{\varphi | p_r M p_r\colon \varphi \in M_*^{\mathbb{G}},\ \varphi \geq 0\}$ is a faithful family of linear positive normal (α_g^r)-invariant functionals, which implies that the α_g^r's are normal.

The following theorem may be regarded as a generalization of Theorem 1.1 from Frigerio and Verri (1982), especially in view of Frigerio and Verri (1982), Remark 2.2.

Theorem 3. Let the α_g's be positive contractions, let ϵ be an ergodic projection, and put

$$\Theta(x) = p_r \epsilon(x) p_r, \qquad x \in M$$

Then Θ is a positive normal (α_g)-invariant projection of norm one onto $(p_r M p_r)^{\mathbb{G}}$—the space of the (α_g^r)-fixed points—and $M_*^{\mathbb{G}} = \{\varphi \in M_*\colon \varphi = \varphi \circ \Theta\}$.

Proof. By (3) we have

$$\Theta^2(x) = p_r \epsilon(p_r \epsilon(x) p_r) p_r = p_r \epsilon^2(x) p_r$$
$$= p_r \epsilon(x) p_r = \Theta(x)$$

so Θ is a projection. The (α_g) invariance of Θ follows from that of ϵ; in particular, $\varphi \circ \Theta \in M^{*\mathbb{G}}$ for $\varphi \in M^*$ and once we have shown that Θ is normal we shall get $\varphi \circ \Theta \in M_*^{\mathbb{G}}$ for $\varphi \in M_*$. On the other hand, if $\varphi \in M_*^{\mathbb{G}}$, then by Lemma 1 we have

$$\varphi(\Theta(x)) = \varphi(p_r \epsilon(x) p_r) = \varphi(\epsilon(x)) = \varphi(x)$$

where the last equality follows from property (iii) in the definition of the ergodic projection.

On account of Thomsen (1985), Theorem 4, for (α_g^r) there is a normal positive unital ergodic projection ϵ^r from $p_r M p_r$ onto $(p_r M p_r)^{\mathbb{G}}$. By (2) we have

$$\alpha_g^r(\Theta(x)) = p_r \alpha_g(p_r \epsilon(x) p_r) p_r = p_r \alpha_g(\epsilon(x)) p_r = \Theta(x)$$

thus

$$\epsilon^r \circ \Theta = \Theta$$

For any $\varphi \in (p_rMp_r)_*$ we have $\varphi \circ \epsilon^r \in (p_rMp_r)_*$, and from what we have already shown it follows that

$$\varphi \circ \epsilon^r \circ \Theta | p_rMp_r = \varphi \circ \epsilon^r$$

Consequently,

$$\epsilon^r \circ \Theta | p_rMp_r = \epsilon^r$$

which gives the equality

$$\epsilon^r = \Theta | p_rMp_r \tag{4}$$

From (3) we obtain

$$\Theta(x) = \Theta(p_rxp_r), \qquad x \in M$$

which together with (4) shows that Θ is a positive normal projection onto $(p_rMp_r)^{\mathrm{G}}$. From (1) and property (iii) in the definition of ergodic projection it follows that $p_r \leq \epsilon(p_r) \leq \epsilon(\mathbf{1}) \leq \mathbf{1}$, so

$$\Theta(\mathbf{1}) = p_r\epsilon(\mathbf{1})p_r = p_r$$

showing that Θ has norm one. ∎

As a corollary to the above theorem we get a result on the structure of the fixed-point space of (α_g).

Corollary 4. Let the α_g's and ϵ be as above. Then

$$p_rM^{\mathrm{G}}p_r = (p_rMp_r)^{\mathrm{G}}$$

In particular, $p_rM^{\mathrm{G}}p_r$ is a JW^*-algebra with identity p_r.

Proof. The first part is an immediate consequence of Theorem 3, while the second follows from the first one and a description of the fixed-point spacer given in Thomsen (1985), Theorem 4. ∎

An important question concerning an ergodic projection is the question of its normality. It turns out that for the α_g's being normal a simple characterization can be given in terms of the recurrent projection. It seems interesting that while normality of the α_g's is essential, their contractivity may be dropped altogether in this case.

Theorem 5. Let the α_g's be positive and normal, and let ϵ be an ergodic projection. Then ϵ is normal if and only if $\epsilon(p_r) = \epsilon(\mathbf{1})$.

Proof. Assume first that $\epsilon(p_r) = \epsilon(\mathbf{1})$. Let

$$\epsilon = \epsilon_n + \epsilon_s$$

be the decomposition of ϵ into its normal and singular parts, respectively (Takesaki, 1979, p. 128). We have, for each $g \in \mathbb{G}$,

$$\epsilon_n + \epsilon_s = \epsilon = \alpha_g \circ \epsilon = \alpha_g \circ \epsilon_n + \alpha_g \circ \epsilon_s$$

and so

$$\epsilon_n - \alpha_g \circ \epsilon_n = \alpha_g \circ \epsilon_s - \epsilon_s$$

Since α_g is normal, it follows that $\alpha_g \circ \epsilon_n$ is normal and $\alpha_g \circ \epsilon_s$ is singular, thus on the left-hand side we have a normal mapping, while on the right-hand side we have a singular one. Consequently, both mappings must be zero, which gives the equalities

$$\epsilon_n = \alpha_g \circ \epsilon_n, \qquad \epsilon_s = \alpha_g \circ \epsilon_s$$

showing that ϵ_n and ϵ_s map M into M^G; in particular,

$$\epsilon \circ \epsilon_n = \epsilon_n, \qquad \epsilon \circ \epsilon_s = \epsilon_s \tag{5}$$

For each $\varphi \in M_*^G$, we have $\varphi = \varphi \circ \epsilon$ by property (iii) in the definition of ergodic projection, and thus

$$\varphi = \varphi \circ \epsilon = \varphi \circ \epsilon_n + \varphi \circ \epsilon_s$$

Since φ and $\varphi \circ \epsilon_n$ are normal and $\varphi \circ \epsilon_s$ is singular, we infer that

$$\varphi \circ \epsilon_s = 0 \tag{6}$$

It has been noticed before that $\{\varphi \in M_*^G: \varphi \geq 0\}$ is a faithful family of linear positive normal functionals on the algebra $p_r M p_r$, which together with equality (6) and positivity of ϵ_s implies that

$$p_r \epsilon_s(x) p_r = 0, \qquad x \in M, \quad x \geq 0$$

In particular, we have

$$\epsilon_s(\mathbf{1}) \leq p_r^\perp$$

which, by virtue of (5), gives

$$\epsilon_s(\mathbf{1}) = \epsilon(\epsilon_s(\mathbf{1})) \leq \epsilon(p_r^\perp) = 0$$

Thus $\epsilon_s(\mathbf{1}) = 0$ and, consequently, $\epsilon_s = 0$, which means that $\epsilon = \epsilon_n$, so ϵ is normal.

Now, let ϵ be normal. For each $\varphi \in M_*$, we have $\varphi \circ \epsilon \in M_*^G$, and by Lemma 1 we get

$$(\varphi \circ \epsilon)(\mathbf{1}) = (\varphi \circ \epsilon)(p_r)$$

showing that $\epsilon(\mathbf{1}) = \epsilon(p_r)$. ■

Remark 3. It can be shown that if the α_g's are positive and normal, then ϵ_n and ϵ_s are (α_g)-invariant projections from M into M^G such that $\epsilon_n \circ \epsilon_s = \epsilon_s \circ \epsilon_n = 0$ and $M_*^G = \{\varphi \circ \epsilon_n: \varphi \in M_*\}$.

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